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NON-LINEAR INTEGRAL EQUATIONS OF THE VOLTERRA TYPE.

By Haig Galajikian.

Introduction.

An existence theorem concerning the non-linear integral equation of the Volterra type

$$u(x) = \phi(x) + \int_{x_0}^{x} f[x, t, u(t)]dt$$

was first given by T. Lalesco,* using the method of successive approximations after Picard. In 1911 this same equation was considered by the author,† independently of T. Lalesco's work. While the general plan of treatment was the same, the region of existence was made less restricted by the introduction of a curved boundary in place of the rectangular one used by T. Lalesco; also a new method for proving the uniqueness of the solution was introduced. The author also established existence theorems for the equation of more general type

$$u(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, u(t)]dt, \int_{x_0}^{x} f_2[x, t, u(t)]dt\right\},$$

the method used, that of successive approximations, applying to a system of n such equations with m arguments involving n unknown functions.

It is evident that the value of a theorem of this sort is considerably increased by the weakening of the conditions imposed on the functions involved and by increasing the size of intervals in which these conditions must hold. Hence, in the first part of this paper, the set of integral equations, mentioned above, is considered, and followed by an effort at an improvement of the existence theorems so far given by widening the interval in which the existence of solution has been established.

It may be that in certain cases it is not only required to know whether or not a solution exists, but whether or not the solution has derivatives; and this naturally comprises the question of continuity. Hence the purpose of the main body of this paper is two-fold. First, it is to show that the solutions given by the existence theorems will, under certain conditions, be continuous functions of both upper and lower limits of the integral arguments appearing in the equations. In other words, it is to show that under the conditions of the problems no discontinuous solution may exist. Secondly, it is to show the existence of first derivatives with respect to the upper and lower limits of integration, in other words with respect to the arguments of the unknown function. Here the methods used necessitate the introduction of such additional conditions as continuity and existence of certain of the partial derivatives of the functions involved in the equations treated.

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* Journal de Mathématiques, series 6, vol. 4 (1908), p. 165.

† Master's thesis, Cornell University, 1912; cf. Bulletin of American Mathematical Society, vol. 19 (1912), pp. 342–346.

The methods used throughout the paper are very simple and straightforward. The establishment of the first derivative of the solution with respect to the upper limit of integration is accomplished in a truly elementary way, void of all difficulties. However, the study of the first derivative of the solution, with respect to the lower limit of integration, necessitates the proof of the existence of a unique solution of an integral equation of a more general type than heretofore considered.

The third part of the paper is devoted to the study of this general non-linear integral equation, under restrictions very similar to those assumed in theorems given in the first part of the paper.

A short section is added at the end to show how the question of first derivatives may successfully be extended to derivatives of any order whatever.

As the forms and lengths of the equations treated are rather cumbersome in certain sections, a notation has been introduced, which, although not claimed to be perfect, does simplify the work to a considerable extent. In what follows an integral equation of the type

$$u(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt, \cdots, \int_{x_0}^{x} f_m[x, t, u(t, x_0)]dt\right\}$$

will be denoted by the equation

$$u(x, x_0) = [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]].$$

The absence of the star over the second bracket indicates that there is only one argument, in the function g, of the form

$$\int_{x_0}^{x} f[x, t, u(t, x_0)]dt.$$

The presence of the star shall mean that the process involved is absolutely the same for any finite number of arguments as for one, and the expression represents the sum of a number of terms similar to the one written. Partial derivatives will be denoted by the symbols

$$_{(x)}[\cdots\cdot] \quad \text{and} \quad _{(x)}[\cdot\cdot{}_{(v)}[\cdot\cdot\cdot]].$$

In order to avoid all possibilities of confusion, we cite here an example from page 183 of the paper. There we have the expression

$$\begin{aligned}
&_{(x)}[x + \theta k,\ [x,\ \overset{x}{\underset{x_0}{t\ ;}}\ u] + \theta\{[x + k,\ \overset{x+k}{\underset{x_0}{t\ ;}}\ u] - [x,\ \overset{x}{\underset{x_0}{t\ ;}}\ u]\}] \\
&\qquad + \frac{1}{k}\{[x + k,\ \overset{x+k}{\underset{x_0}{t\ ;}}\ u] - \{x,\ \overset{x}{\underset{x_0}{t\ ;}}\ u]\}_{(v_m)}[x + \theta k,\ [x,\ \overset{x}{\underset{x_0}{t\ ;}}\ u] \\
&\qquad\qquad + \theta\{[x + k,\ \overset{x+k}{\underset{x_0}{t\ ;}}\ u] - [x,\ \overset{x}{\underset{x_0}{t\ ;}}\ u]\}]
\end{aligned}$$

which written out in full for two arguments v_1, v_2 would be

$$\begin{aligned}
&g_x\Big\{x + \theta k,\ \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt + \theta\Big[\int_{x_0}^{x+k} f_1[x + k, t, u(t, x_0)]dt \\
&\quad - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big],\ \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt \\
&\quad + \theta\Big[\int_{x_0}^{x+k} f_2[x + k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big]\Big\} \\
&\quad + \frac{1}{k}\Big[\int_{x_0}^{x+k} f_1[x + k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big] \cdot g_{v_1}\Big\{x + \theta k, \\
&\quad \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt + \theta\Big[\int_{x_0}^{x+k} f_1[x + k, t, u(t, x_0)]dt \\
&\quad - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big],\ \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt \\
&\quad + \theta\Big[\int_{x_0}^{x+k} f_2[x + k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big]\Big\} \\
&\quad + \frac{1}{k}\Big[\int_{x_0}^{x+k} f_2[x + k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big] \\
&\qquad\qquad \times g_{v_2}\,\{\text{similar terms as in } g_{v_1}\}.
\end{aligned}$$

This example will show the advantage of the notation.

1. The System of Integral Equations

$$\begin{aligned} u_1(x, x_0) &= {}^1[x, [x, t \overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]], \\ u_2(x, x_0) &= {}^2[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]], \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \\ u_n(x, x_0) &= {}^n[x, [x, t \overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]]. \end{aligned} \tag{1}$$

THEOREM. *Let* ${}^1[x, v_1, \cdots, v_m], \cdots, {}^n[x, v_1, \cdots, v_m]$ *be continuous and satisfy Lipschitz conditions for the arguments* $v_1, v_2, \cdots, v_m$, *namely*

$$|{}^i[x, v_1 + \mu_1, v_2 + \mu_2 \cdots, v_m + \mu_m] - {}^i[x, v_1, v_2, \cdots, v_m]| \leqq A \sum_{k=1}^{m} |\mu_k|$$
$$(i = 1, 2, \cdots n),$$

in the region $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $|v_2| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$. *Let* $[x, t; u_1, u_2, \cdots, u_n]$ *be continuous functions, and satisfy Lipschitz conditions for the arguments,* $u_1, u_2, \cdots, u_n$, *namely*

$$|[x, t; u_1 + \nu_1, u_2 + \nu_2, \cdots, u_n + \nu_n] - [x, t; u_1, u_2, \cdots, u_n]| \leqq B \sum_{i=1}^{n} |\nu_i|$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u_i - {}^i[x, 0, 0, \cdots, 0]| \leqq \gamma$, $(i = 1, 2, \cdots, n)$; *furthermore let*

$$|[x, t; u_1, u_2, \cdots, u_n]| \leqq M.$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{M}, \qquad \rho \leqq \frac{\gamma}{mAM},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one system of continuous functions,*

$$u_1(x, x_0), \qquad u_2(x, x_0), \qquad \cdots, \qquad u_n(x, x_0),$$

satisfying the system of integral equations (1).

Define the functions

$$\begin{aligned} u_{i0}(x, x_0) &= {}^i[x, 0, 0, \cdots, 0] \qquad (i = 1, 2, \cdots, n] \\ u_{ik}(x, x_0) &= {}^i[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u_{1k-1}, u_{2k-1}, \cdots, u_{nk-1}]] \\ &\qquad (i = 1, 2, \cdots, n), \ (k = 1, 2, \cdots). \end{aligned} \tag{2}$$

It is easily seen that at each stage of the approximation the conditions

$$|u_{ik}(x, x_0) - {}^i[x, 0, 0, \cdots, 0]| \leqq \gamma \quad (i = 1, 2, \cdots, n),$$

$$|[x, t \overset{x}{\underset{x_0}{;}} u_{1k}, u_{2k}, \cdots, u_{nk}]| \leqq \beta \qquad (k = 1, 2, \cdots),$$

are satisfied when $|x - x_0| \leqq \rho$, and hence the next following set of approximation will have a meaning. Evidently each $u_{ik}(x, x_0)$ is continuous for $|x - x_0| \leqq \rho$.

We shall prove that the sequences

$$u_{1k}(x, x_0), \qquad u_{2k}(x, x_0), \qquad \cdots, \qquad u_{nk}(x, x_0),$$

approach, uniformly, the limits

$$u_1(x, x_0), \qquad u_2(x, x_0), \qquad \cdots, \qquad u_n(x, x_0).$$

To this end we write,

$$u_{ik}(x, x_0) - u_{ik-1}(x, x_0) = U_{ik}(x, x_0),$$

so that

$$u_{ik}(x, x_0) = u_{i0}(x, x_0) + U_{i1}(x, x_0) + \cdots + U_{ik}(x, x_0),$$
$$(i = 1, 2, \cdots, n), \ (k = 1, 2, \cdots).$$

The Lipschitz conditions on ${}^i[x, v_1, v_2, \cdots, v_n]$ and $[x, t; u_1, u_2, \cdots, u_n]$ give us,

$$\begin{aligned}
|U_{ik}(x, x_0)| &\leqq \frac{mnAB}{n}\left|\int_{x_0}^{x}\sum_{i=1}^{n}|U_{ik-1}(t, x_0)|dt\right| \\
&\leqq \frac{m^2n^2A^2B^2}{n}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\sum_{i=1}^{n}|U_{ik-2}(t_1, x_0)|dt_1\right|dt\right| \\
\cdots\quad &\leqq \frac{m^{k-1}n^{k-1}A^{k-1}B^{k-1}}{n}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\right|\cdot\int_{x_0}^{t_1}\right| \\
&\qquad\cdots\left|\int_{x_0}^{t_{k-2}}\sum_{i=1}^{n}|U_{i1}(t_{k-1}, x_0)|dt_{k-1}\right|\cdots\left|dt_1\right|dt\Bigg| \\
&\leqq \frac{m^kn^kA^kB^k}{nB}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\right|\cdots\left|\int_{x_0}^{t_{k-2}}|M|t_{k-1} - x_0|dt_{k-1}\right|\cdots\left|dt_1\right|dt\right| \\
&\leqq \left(\frac{M}{nB}\right)\cdot\left[\frac{(mnAB\rho)^k}{k!}\right] \quad \text{when} \quad |x - x_0| \leqq \rho.
\end{aligned}$$

Since the series of constants

$$\sum_{k=1}^{\infty}\left[\frac{(mnAB\rho)^k}{k!}\right]$$

converges, we see that the series

$$u_{i0}(x, x_0) + U_{i1}(x, x_0) + U_{i2}(x, x_0) + \cdots + U_{ik}(x, x_0) + \cdots$$

converge uniformly, $|x - x_0| \leqq \rho$; if we represent their values by

$$u_1(x, x_0), \quad u_2(x, x_0), \quad \cdots, \quad u_n(x, x_0),$$

we have

$$\operatorname*{Lim}_{k=\infty} u_{1k}(x, x_0) = u_1(x, x_0), \qquad \cdots, \qquad \operatorname*{Lim}_{k=\infty} u_{nk}(x, x_0) = u_n(x, x_0)$$

uniformly, $|x - x_0| \leqq \rho$. By reference to (2) we see that these limits satisfy the system (1).

That there is only one such set of continuous functions, satisfying the system (1), may be seen as follows. Suppose there were two sets of continuous functions $u_i(x, x_0)$ and $z_i(x, x_0)$ satisfying the set (1), and put

$$u_i(x, x_0) - z_i(x, x_0) = w_i(x, x_0).$$

If we now write the set of equations (1) for $u_i(x, x_0)$ and $z_i(x, x_0)$, and subtract, we see from conditions on the functions ${}^i[x, v_1, v_2, \cdots, v_m]$ and $[x, t; u_1, u_2, \cdots, u_n]$, that

$$|w_i(x, x_0)| \leqq mAB\left|\int_{x_0}^{x} |w_i(t, x_0)|dt\right|, \tag{3}$$

and if we write $|w_i(x, x_0)| \leqq W$, we have by successive applications of (3)

$$|w_i(x_1x_0)| \leqq mnABW|x - x_0|, \qquad |w_i(x, x_0)| \leqq m^2n^2A^2B^2W\frac{(x - x_0)^2}{2!},$$

and in general

$$|w_i(x, x_0)| \leqq W\frac{[mnAB|x - x_0|]^k}{k!}.$$

The expression on the right has the limit zero as k becomes infinite, therefore

$$w_i(x, x_0) = 0, \qquad \text{or} \qquad u_i(x, x_0) = z_i(x, x_0).$$

2. **Removal of the condition**

$$|x - x_0| \leqq \rho \leqq \frac{\gamma}{mAM}.$$

In the preceeding article the existence of a set of continuous functions, satisfying the given set of integral equations, was shown in a region smaller than the one in which continuity and the Lipschitz conditions were imposed on the functions ${}^i[x, v_1, v_2, \cdots, v_m]$ and $[x, t; u_1, u_2, \cdots, u_n]$. More particularly, we had

$$|x - x_0| \leqq \rho; \qquad \rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{M}, \qquad \rho \leqq \frac{\gamma}{mAM}.$$

The removal of any one of these condition, on our interval, will undoubtedly add to the value of the theorem. The third of the conditions on ρ may be removed by a simple procedure, depending on the part of the existence theorem of the above section dealing with the uniqueness of the solution. As this condition for a system of equations of the type (1) is identically the same as for one equation of the form

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u]],$$

and as the argument employed is identically the same as for an equation

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u]],$$

where the third condition on ρ is

$$\rho \leqq \frac{\gamma}{AM},$$

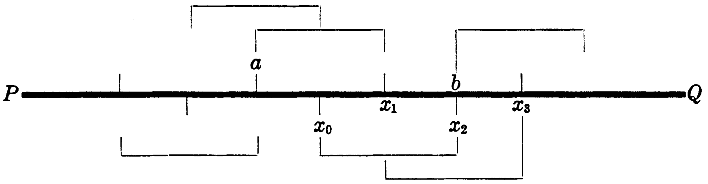
we shall treat this last equation here.

Let the line PQ represent the interval

$$|x - x_0| \leqq \min\left(\alpha, \frac{\beta}{M}\right),$$

and subdivide this line so that

$$x_\lambda - x_{\lambda-1} = \frac{\gamma}{AM}.$$



In any one of the subintervals $|x_\lambda - x_{\lambda-1}| \leqq \gamma/AM$ a solution of

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_{\lambda-1}}{;}} u]]$$

exists and is unique. Let us call $u^0(x, x_0)$ the solution of

$$u^0(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u^0]],$$

for $|x - x_0| \leqq x_1 - x_0$; it exists and is unique. Now consider

$$(1) \qquad u^{(1)}(x, x_0) = [x, [x, \overset{x_1}{\underset{x_0}{t}} ; u^0] + [x, \overset{x}{\underset{x_1}{t}} ; u^{(1)}]] = [x, [x, \overset{x}{\underset{x_1}{t}} ; u^{(1)}]]^{(1)},$$

an integral equation whose solution exists for $|x - x_1| \leqq x_1 - x_0$, and is unique, for

$$|[x, v + \mu]^{(1)} - [x, v]^{(1)}| \leqq A\mu,$$

and for a small enough subinterval

$$|[x, t \underset{x_0}{\overset{x}{;}} u^0] + v| \leqq \beta,$$

or

$$|v| \leqq \beta - M(x_1 - x_0),$$

and

$$|x - x_0| \leqq \alpha, \qquad |t - x_0| \leqq \alpha.$$

This may be seen from the approximations

$$u_0^{(1)}(x, x_0) = [x, 0]^{(1)},$$

$$u_\nu^{(1)}(x, x_0) = [x, [x, t \underset{x_1}{\overset{x}{;}} u_{\nu-1}^{(1)}]]^{(1)},$$

where

$$|[x, t \underset{x_1}{\overset{x}{;}} u_\nu^{(1)}]| \leqq \beta - M(x_1 - x_0),$$

or

$$M|x - x_1| \leqq \beta - M(x_1 - x_0),$$

which is evidently true for $x_1 \leqq x \leqq x_2$ and consequently also for $x_0 \leqq x \leqq x_1$, and we also have

$$|u_\nu^{(1)}(x, x_0) - [x, 0]^{(1)}| \leqq \gamma.$$

Suppose now $x_0 < x < x_1$, then

$$u^{(0)}(x, x_0) = [x, [x, t \underset{x_0}{\overset{x}{;}} u^{(0)}]] = [x, [x, t \underset{x_0}{\overset{x_1}{;}} u^{(0)}] + [x, t \underset{x_1}{\overset{x}{;}} u^{(0)}]] = [x, [x, t \underset{x_0}{\overset{x}{;}} u^{(0)}]]^{(1)}.$$

Therefore $u^0(x, x_0)$ is a solution of (1) for $x_0 < x < x_1$. But the solution of (1) is unique for $|x - x_1| \leqq (x_1 - x_0)$; it follows therefore that

$$u^{(0)}(x, x_0) \equiv u^{(1)}(x, x_0) \quad \text{for} \quad x_0 \leqq x \leqq x_1.$$

Therefore there is one and only one solution of our original integral equation in the extended interval (a, b). Continuing this process, we extend this solution to the interval PQ.

3. Continuity of the Solution.

THEOREM: *Given the integral equation*

$$u(x, x_0) = [x, [x, t \underset{x_0}{\overset{x}{;}} u(t, x_0)]]; \tag{1}$$

let $[x, v_1, v_2, \cdots, v_m]$ *be continuous and satisfy for all its arguments** *the Lipschitz conditions*

$$|[x+\mu_0, v_1+\mu_1, v_2+\mu_2, \cdots, v_m+\mu_m] - [x, v_1, v_2, \cdots, v_m]| \leqq A\sum_{k=0}^{m}|\mu_k|$$

in the region $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $|v_2| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$. *Let* $[x, t; u]$ *be continuous functions, and let them satisfy, for the arguments* x *and* u, *the Lipschitz conditions,*

$$|[x+\theta, t; u+\eta] - [x, t; u]| \leqq B\{|\theta| + |\eta|\},$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - [x, 0, \cdots, 0]| \leqq \gamma$. *Furthermore let*

$$|[x, t; u]| \leqq M.$$

Then if there is any solution whatever, in the interval $|x - x_0| \leqq \alpha$, *it is continuous in* x *and* x_0. In particular, then, the unique solution of the equation (1) in $|x - x_0| \leqq \rho$, $\rho \leqq \min(\alpha, \beta/M)$, is continuous in x and x_0 in this reduced interval. This also applies to the set of solutions of the system of equations of § 1.

We want to show that, given a positive number ϵ, however small, we can make the absolute value of the difference

$$u(x+k, x_0+h) - u(x, x_0)$$

become and remain less than ϵ, by choosing $|k|$ and $|h|$ small enough.

Write

$$\begin{aligned}(2)\quad &|u(x+k, x_0+h) - u(x, x_0)| \\ &\leqq \{|u(x+k, x_0+h) - u(x, x_0+h)| + |u(x, x_0+h) - u(x, x_0)|\},\end{aligned}$$

the two parts of which may be made to become and remain less than $\epsilon/2$, respectively, for $|k|$ and $|h|$ small enough.

To this end we call

$$u_1(x, x_0) = [x, [x, \overset{x}{\underset{x_0+h}{t\,;}}\, u(t, x_0+h)]],$$

$$u(x, x_0) = [x, [x, \overset{x}{\underset{x_0}{t\,;}}\, u(t, x_0)]],$$

and

$$z(x, x_0) = u_1(x, x_0) - u(x, x_0).$$

Breaking up (2) into its two parts we have,

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[*] Notice here that an additional Lipschitz condition on the argument x is necessary whereas that was not at all necessary for the existence theorem in the form it appears in in this paper. See Bulletin of the American Mathematical Society, vol. 19 (1912), p. 345.

$$(a)\qquad \begin{aligned}&|u(x+k, x_0+h) - u(x, x_0+h)|\\ &\qquad = |[x+k, \overset{*}{[}x+k, t \overset{x+k}{\underset{x_0+h}{;}} u_1]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u_1]]|;\end{aligned}$$

$$(b)\qquad |u(x, x_0+h) - u(x, x_0)| = |[x, \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u_1]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]]|.$$

Taking (a) we have

$$\begin{aligned}&|u(x+k, x_0+h) - u(x, x_0+h)|\\ &\qquad = |[x+k, \overset{*}{[}x+k, t \overset{x}{\underset{x_0+h}{;}} u_1] + \overset{*}{[}x+k, t \overset{x+k}{\underset{x}{;}} u_1]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u_1]]|,\end{aligned}$$

which, because of the Lipschitz condition on $[x, v_1, \cdots, v_m]$, is

$$\leqq A\{|k| + |\overset{*}{[}x+k, t \overset{x}{\underset{x_0+h}{;}} u_1] - \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u_1]| + |\overset{*}{[}x+k, t \overset{x+k}{\underset{x}{;}} u_1]|\}$$

$$\leqq A\left\{|k| + \left|\int_{x_0+h}^{x} |\overset{*}{[}x+k, t; u_1] - \overset{*}{[}x, t; u_1]|dt\right| + \left|\int_{x}^{x+k} |\overset{*}{[}x+k, t; u_1]|dt\right|\right\},$$

which, because of the Lipschitz condition on $\overset{*}{[}x, t; u]$, is

$$\leqq A\left\{|k| + mB|k|\left|\int_{x_0+h}^{x} dt\right| + mM|k|\right\}$$

$$\leqq A\{|k| + mB|k(x-x_0-h)| + mM|k|\} = A|k|\{1 + mB\alpha + mM\},$$

where

$$|x - x_0 - h| = \alpha.$$

Calling now $A\{1 + mB\alpha + mM\} = P$, where P is a finite constant, we get

$$|u(x+k, x_0+h) - u(x, x_0+h)| \leqq P|k|.$$

Treating (b) in the same manner, we have

$$\begin{aligned}|z(x, x_0)| &= |[x, \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u_1]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]]|\\ &= |[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u_1] - \overset{*}{[}x, t \overset{x_0+h}{\underset{x_0}{;}} u_1]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]]|\\ &\leqq A\left\{\left|\int_{x_0}^{x} |\overset{*}{[}x, t; u_1] - \overset{*}{[}x, t; u]|dt\right| + \left|\int_{x_0}^{x_0+h} |\overset{*}{[}x, t]; u|_1\, dt\right|\right\}\\ &\leqq A\left\{mB\left|\int_{x_0}^{x} |z|dt\right| + mM|h|\right\} = \lambda|w(x, x_0)| + \mu|h|,\end{aligned}$$

where

$$\lambda = mAB, \qquad mAM = \mu, \qquad \int_{x_0}^{x} |z(t, x_0)|dt = w(x, x_0).$$

Thus we have the differential inequality

$$|w'(x, x_0)| \leqq \lambda |w(x, x_0)| + \mu |h|,$$

from which follows* that $w(x, x_0)$ satisfies the inequality

$$|w(x, x_0)| \leqq |w(x_0, x_0)| e^{\lambda|x-x_0|} + \frac{\mu|h|}{\lambda}(e^{\lambda|x-x_0|} - 1).$$

However,

$$w(x_0, x_0) = 0,$$

therefore

$$|w(x, x_0)| \leqq |h| \frac{\mu}{\lambda}(e^{\lambda|x-x_0|} - 1),$$

and finally from

$$w'(x, x_0) = |z(x, x_0)| \leqq \lambda |w(x, x_0)| + \mu |h|.$$

we have

$$|z(x, x_0)| \leqq \mu |h| e^{\lambda|x-x_0|}.$$

Combining this with the result of (*a*), we have

$$|u(x + k, x_0 + h) - u(x, x_0)| < \epsilon,$$

for $|k|$ and $|h|$ sufficiently small.

4. First Partial Derivatives of the Solution.

THEOREM: *Given the integral equation*

$$u(x, x_0) = [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]]; \tag{1}$$

let $[x, v_1, v_2, \cdots, v_m]$ *be continuous and satisfy Lipschitz conditions for the arguments* $v_1, v_2, \cdots, v_m$, *in the region* $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta, \cdots, |v_m| \leqq \beta$. *Let the first partial derivatives of* $[x, v_1, v_2, \cdots, v_m]$ *with respect to all its arguments exist and be continuous. Let* $\overset{*}{[}x, t; u]$ *be continuous functions and satisfy Lipschitz conditions for the argument* u *in the region* $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - [x, v_1, \cdots, v_m]| \leqq \gamma$. *Let the first partial derivatives of* $\overset{*}{[}x, t; u]$ *with respect to arguments* x *and* u *exist and be continuous in the same region. Furthermore let*

$$|\overset{*}{[}x, t; u]| \leqq M,$$

and

$$|[x, \overset{*}{[}x, t; u]]| \leqq N.$$

Then if $\rho \leqq \alpha$, $\rho \leqq \beta/M$, *the unique solution of* (1) *has first derivatives with respect to* x *and* x_0.†

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* See for instance Bliss, Princeton Colloquium (1909), p. 88.

† It may be observed that the conditions, required for the existence of the first partial derivative with respect to x alone, are very much weaker than the ones stated above for both partial derivatives.

We want to show the existence of the following limits

$$(a) \qquad \operatorname*{Lim}_{k=0}\left[\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k}\right],$$

and

$$(b) \qquad \operatorname*{Lim}_{h=0}\left[\frac{u(x,\,x_0+h)-u(x,\,x_0)}{h}\right].$$

Taking (a) first, we have

$$\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k} = \frac{1}{k}\{[x+k,\ \overset{*}{[}x+k,\ t \overset{x+k}{\underset{x_0}{;}} u]] - [x,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]]\}$$

$$= \frac{1}{k}\{[x+k,\ \overset{*}{[}x+k,\ t \overset{x}{\underset{x_0}{;}} u] + \overset{*}{[}x+k,\ t \overset{x+k}{\underset{x}{;}} u]] - [x,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]]\},$$

which, by the laws of the mean for differentials and integrals, becomes

$${}_{(x)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + \theta\{\overset{*}{[}x+k,\ t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]\}]$$

$$+\frac{1}{k}\{\overset{*}{[}x+k,\ t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]\}\,{}_{(v_m)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+\theta\{\overset{*}{[}x+k,\ t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]\}] = {}_{(x)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+\theta\{\overset{*}{[}x+k,\ t \overset{x}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + \overset{*}{[}x+k,\ t \overset{x+k}{\underset{x}{;}} u]\}$$

$$+\frac{1}{k}\{\overset{*}{[}x+k,\ t \overset{x}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + \overset{*}{[}x+k,\ t \overset{x+k}{\underset{x}{;}} u]\}\,{}_{(v_m)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+\theta\{\overset{*}{[}x+k,\ t \overset{x}{\underset{x_0}{;}} u] - \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + \overset{*}{[}x+k,\ t \overset{x+k}{\underset{x}{;}} u]\}],$$

and

$$\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k} = {}_{(x)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + k\theta\{{}_{(x)}\overset{*}{[}x+\theta_1 k,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+ \overset{*}{[}x+k,\ x+\theta_2 k,\ u(x+\theta_2 k,\ x_0)]\}] + \{{}_{(x)}\overset{*}{[}x+\theta_1 k,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+ \overset{*}{[}x+k,\ x+\theta_2 k,\ u(x+\theta_2 k,\ x_0)]\} \cdot {}_{(v_m)}[x+\theta k,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]$$

$$+ k\theta\{{}_{(x)}\overset{*}{[}x+\theta_1 k,\ t \overset{x}{\underset{x_0}{;}} u] + \overset{*}{[}x+k,\ x+\theta_2 k,\ u(x+\theta_2 k,\ x_0)]\}],$$

where θ, θ_1, θ_2 are positive proper fractions. Passing to the limit, we have

$$\frac{\partial u(x,\,x_0)}{\partial x} = {}_{(x)}[x,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]] + \{{}_{(x)}\overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u] + [x,\ x,\ u(x,\ x_0)]\}\ {}_{(v_m)}[x,\ \overset{*}{[}x,\ t \overset{x}{\underset{x_0}{;}} u]].$$

Again we have,

$$\begin{aligned}\frac{u(x, x_0+h)-u(x, x_0)}{h} &= \frac{1}{h}\{[x, \overset{*}{[}x, t \overset{x}{\underset{x_0+h}{;}} u(t, x_0+h)]] - [x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]]\} \\ &= \frac{1}{h}\{\overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u(t, x_0+h)] - \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] - \overset{*}{[}x, t \overset{x_0+h}{\underset{x_0}{;}} u(t, x_0+h)]\} \\ &\times {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] + \theta\{\overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u(t, x_0+h)] - \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] - \overset{*}{[}x, t \overset{x_0+h}{\underset{x_0}{;}} u(t, x_0+h)]\}] \\ &= \frac{1}{h}\left\{\int_{x_0}^{x} {}_{(u)}\overset{*}{[}x, t\,; u(t, x_0+\theta_1 h)](u(t, x_0+h)-u(t, x_0))dt\right. \\ &\left. - h\overset{*}{[}x, x_0+\theta_2 h, u(x_0+\theta_2 h, x_0+\theta_1 h)]\right\} \cdot {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] \\ &+ \theta\left\{\int_{x_0}^{x} {}_{(u)}\overset{*}{[}x, t\,; u(t, x_0+\theta_1 h)](u(t, x_0+h)-u(t, x_0))dt\right. \\ &\left. - h\overset{*}{[}x, x_0+\theta_2 h, u(x_0+\theta_2 h, x_0+\theta_1 h)]\right\},\end{aligned}$$

and taking the limit of both sides, we have

$$\begin{aligned}&\lim_{h=0}\left[\frac{u(x, x_0+h)-u(x, x_0)}{h}\right] \\ &\qquad = \left\{\int_{x_0}^{x} {}_{(u)}[x, t\,; u] \cdot \lim_{h=0}\left[\frac{u(t, x_0+h)-u(t, x_0)}{h}\right]dt\right. \\ &\qquad\qquad \left. - \overset{*}{[}x, x_0, 0]\right\} {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]].\end{aligned}$$

Assuming that the limit, appearing on either side, exists, and denoting it by

$$z(x, x_0) = \frac{\partial u(x, x_0)}{\partial x_0},$$

we have

$$u(x, x_0) = -\int_{x_0}^{x} z(x, \xi)d\xi,$$

and substituting above, we have

$$\begin{aligned}(1)' \quad z(x, x_0) &= \left\{ {}_{(u)}\overset{*}{\left[}\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z(t, \xi)d\xi\right) z(t, x_0)\right]\right. \\ &\qquad \left. - \overset{*}{[}x, x_0, 0]\right\} \cdot {}_{(v_m)}\left[x, \overset{*}{\left[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z(t, \xi)d\xi\right]\right],\end{aligned}$$

which is an integral equation of a type not considered heretofore.

Now if the assumption about the limit of

$$\frac{u(x,\, x_0 + h) - u(x,\, x_0)}{h},$$

as h becomes indefinitely small, is true, a formal integration of $(1)^1$, with respect to x_0, ought to lead us back to equation (1). If this does happen, then to make the theorem complete, we must prove the existence of a unique solution of $(1)^1$ under the conditions of the problem. But $(1)^1$, as it stands, is the result of a formal differentiation of (1), with respect to x_0, and we therefore proceed to the existence theorem of a unique solution of $(1)^1$ which we write in the more general form:

$$\begin{aligned} z(x,\, x_0) = \Bigg\{ \overset{*}{\Bigg[} \Bigg(x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z(t,\, \xi)\,d\xi \Bigg) z \Bigg] \\ - \overset{*}{[} x,\, x_0,\, 0] \Bigg\} \cdot {}^{i}\Bigg[x,\, \overset{*}{\Bigg[} \Bigg(x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z(t,\, \xi)\,d\xi \Bigg) \Bigg] \Bigg] \qquad (i = 1,\, 2,\, \cdots,\, m). \end{aligned} \tag{2}$$

Define the functions

$$z_0(x,\, x_0) = - \overset{*}{[} x,\, x_0,\, 0]^{(i)} [x,\, 0,\, 0],$$

$$\begin{aligned} z_1(x,\, x_0) = \Bigg\{ \overset{*}{\Bigg[} \Bigg(x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t,\, \xi)\,d\xi \Bigg) z_0 \Bigg] - \overset{*}{[} x,\, x_0,\, 0] \Bigg\} \\ \times {}^{i}\Bigg[x,\, \overset{*}{\Bigg[} x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t,\, \xi)\,d\xi \Bigg] \Bigg], \end{aligned}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

$$\begin{aligned} z_n(x,\, x_0) = \Bigg\{ \overset{*}{\Bigg[} \Bigg(x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_{n-1}(t,\, \xi)\,d\xi \Bigg) z_{n-1} \Bigg] - \overset{*}{[} x,\, x_0,\, 0] \Bigg\} \\ \times {}^{i}\Bigg[x,\, \overset{*}{\Bigg[} x,\, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_{n-1}(t,\, \xi)\,d\xi \Bigg] \Bigg], \end{aligned}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

and such that

$$\int_{x_0}^{x} z_0(x,\, \xi)\,d\xi = \int_{x_0}^{x} - \overset{*}{[} x,\, \xi,\, 0]^{(i)} [x,\, 0,\, 0]\,d\xi = [x,\, 0,\, 0] \qquad \text{of (1).}$$

It is evident that at each stage of the approximation the conditions

$$\left| - \int_{x_0}^{x} z_n(x,\, \xi)\,d\xi - [x,\, 0,\, 0] \right| \leqq \gamma \qquad (n = 1,\, 2,\, \cdots),$$

and

$$\left|\overset{*}{\Bigg[} x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_n(t, \xi)\,d\xi \Bigg]\right| \leqq \beta$$

are satisfied for $|x - x_0| \leqq \rho$.

We need to show the uniform convergence, toward a definite limit, of the above defined sequence of approximate expressions—a limit that will satisfy (2), as is evident from the mode of the approximations.

To this end we write

$$z_n(x, x_0) - z_{n-1}(x, x_0) = Z_n(x, x_0) \qquad (n = 1, 2, \cdots),$$

so that

$$z_n(x, x_0) = z_0(x, x_0) + \sum_{i=1}^{n} Z_i(x, x_0) \qquad (n = 1, 2, \cdots).$$

Now if we let N be the larger of the maximum absolute values of ${}^{(i)}[x, v_1, \cdots, v_m]$; M, the larger of the maximum absolute values of

$$\overset{*}{\Bigg[}\Bigg(x, t\,;\, -\int_{x_0}^{t} z_n(t, \xi)\,d\xi\Bigg) z_n\Bigg],$$

we have

$$\begin{aligned}
|Z_1(x, x_0)| &= |z_1(x, x_0) - z_0(x, x_0)| \leqq \Bigg|\Bigg\{\overset{*}{\Bigg[}\Bigg(x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg) z_0\Bigg] \\
&\quad - \overset{*}{[}x, x_0, 0]\Bigg\}\,{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg] - \{-\overset{*}{[}x, x_0, 0)]^{(i)}[x, 0, 0]\}\Bigg| \\
&\leqq \Bigg|\Bigg\{\overset{*}{\Bigg[}\Bigg(x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg) z_0\Bigg] - \overset{*}{[}x, x_0, 0]\Bigg\}\,{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg] \\
&\quad - \{-[\overset{*}{x}, x_0, 0]\}\,{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg]\Bigg| \\
&\quad + \Bigg|\{-\overset{*}{[}x, x_0, 0]\}\,{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg] - \{-\overset{*}{[}x, x_0, 0]^{(i)}[x, 0, 0]\Bigg| \\
&\leqq \Bigg\{\Bigg|\overset{*}{\Bigg[}\Bigg(x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg) z_0\Bigg]\Bigg|\,\Bigg|{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg]\Bigg| \\
&\quad + |\{-\overset{*}{[}x, x_0, 0]|\,\Bigg|{}^{i}\Bigg[x, \overset{*}{\Bigg[}x, t \overset{x}{\underset{x_0}{;}} - \int_{x_0}^{t} z_0(t, \xi)\,d\xi\Bigg]\Bigg] - {}^{(i)}[x, 0, 0]\Bigg| \\
&\leqq mN^2M|x - x_0| + m^2AM^2(x - x_0) = m[N^2M + mAM^2]|x - x_0|,
\end{aligned}$$

which we call ${}_{P_1}|x - x_0|$.

Now

$$|Z_2(x, x_0)| \leqq \Big|\Big\{\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big) z_1\Big]$$

$$- \overset{*}{[}x, x_0, 0]\Big\}\, {}^{i}\Big[x, \Big[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big]\Big] - \Big\{\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big]$$

$$- \overset{*}{[}x, x_0, 0]\Big\}\, {}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big]\Big]\Big| + \Big|\Big\{\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big]$$

$$- \overset{*}{[}x, x_0, 0]\Big\}\, {}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big]\Big] - \Big\{\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big]$$

$$- \overset{*}{[}x, x_0, 0]\Big\}\, {}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big]\Big]\Big| \leqq \Big|\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big) z_1\Big]$$

$$- \overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big]\Big|\,\Big|{}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big]\Big]\Big|$$

$$+ \Big|\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big] - \overset{*}{[}x, x_0, 0]\Big|\,\Big|{}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big]\Big]$$

$$- {}^{i}\Big[x, \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big]\Big]\Big| \leqq N\Big|\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big) z_1\Big]$$

$$- \overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big) z_0\Big]\Big| + \Big|\overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big) z_0\Big]$$

$$- \overset{*}{\Big[}\Big(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big) z_0\Big]\Big| + m\{NM|x - x_0| + M\}A$$

$$\times \Big|\overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_1(t, \xi)d\xi\Big] - \overset{*}{\Big[}x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\Big]\Big|$$

$$\leqq mNMP_1\frac{|x - x_0|^2}{2!} + \{mN^2B + m^2(NM|x - x_0| + M)AB\}\frac{P_1|x - x_0|^3}{3!}.$$

or

$$|Z_2(x, x_0)| \leqq mMNP_1\frac{|x - x_0|^2}{2} + \Big\{\frac{P_1B}{M} + m^2MNAB|x - x_0|\Big\}\frac{P_1|x - x_0|^3}{3}.$$

Similarly it may be shown that

$$|Z_3(x, x_0)| \leqq mMNP_2\frac{|x - x_0|}{3} + \Big\{\frac{P_1B}{M} + m^2MNAB|x - x_0|\Big\}P_2\frac{|x - x_0|^2}{4},$$

and in general

$$|Z_n(x, x_0)| \leqq mMNP_{n-1}\frac{|x-x_0|}{n} + \left\{\frac{P_1B}{M} + m^2MNAB|x-x_0|\right\}\frac{P_{n-1}|x-x_0|^2}{n+1},$$

where P_n is the quantity such that

$$|Z_n(x, x_0)| < P_n.$$

Consequently, for

$$|x - x_0| \leqq \rho, \qquad \frac{P_n}{P_{n-1}} \leqq mMN\frac{\rho}{n} + \left(\frac{P_1B}{M} + m^2MNAB\rho\right)\frac{\rho^2}{n+1},$$

so that

$$\operatorname*{Lim}_{n=\infty}\frac{P_n}{P_{n-1}} = 0;$$

therefore,

$$\operatorname*{Lim}_{n=\infty} z_n(x, x_0) = z(x, x_0)$$

uniformly, and this limit $z(x, x_0)$ is a solution of (2).

That this solution is unique may be seen as follows. Suppose if possible there were two continuous solutions $z(x, x_0)$ and $z'(x, x_0)$ and call

$$w(x, x_0) = z(x, x_0) - z'(x, x_0).$$

It is evident that $w(x, x_0)$ is limited. If we write

$$|w(x, x_0)| \leqq W,$$

we can show, by methods similar to those used in the main body of the existence theorem, that

$$\begin{aligned} |w(x, x_0)| &\leqq mNM\left|\int_{x_0}^{x}|z(t, x_0) - z'(t, x_0)|dt\right| \\ &\quad + \left[\frac{PB}{M} + Q|x - x_0|\right]\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}|z(t, \xi) - z'(t, \xi)|d\xi\right|dt\right|, \end{aligned} \tag{3}$$

where

$$P = mMN^2 + m^2M^2A \quad \text{and} \quad Q = m^2MNAB.$$

From (3) we have

$$|w(x, x_0)| \leqq \left\{mMN + \frac{PB}{M}|x - x_0| + Q|x - x_0|^2\right\}W|x - x_0|.$$

Substituting this last result in (3), we get

$$
\begin{aligned}
|w(x,\,x_0)| &\leqq mMN\,\frac{|x-x_0|^2}{2}\left[mMNW+\frac{PBW}{M}\,|x-x_0|+QW|x-x_0|^2\right]\\
&\quad+\left[\frac{PB}{M}+Q|x-x_0|\right]\left[mMNW+\frac{PBW}{M}\,|x-x_0|+QW|x-x_0|^2\right]\frac{|x-x_0|^3}{3!}\\
&=\left[mMNW+\frac{PBW}{M}\,|x-x_0|+QW|x-x_0|^2\right]\left[mNM\,\frac{|x-x_0|^2}{2!}\right.\\
&\quad\left.+\left\{\frac{PB}{M}+Q|x-x_0|\right\}\right]\frac{|x-x_0|^3}{3!}\leqq\left[mMN+\frac{PB}{M}\,|x-x_0|\right.\\
&\quad\left.+Q|x-x_0|^2\right]^2\frac{W|x-x_0|^2}{2!}\,.
\end{aligned}
$$

Repeating this process, we have

$$
|w(x,\,x_0)|\leqq\left[mMN+\frac{PB}{M}\,|x-x_0|+Q|x-x_0|^2\right]^3\frac{W|x-x_0|^3}{3!}\,,
$$

and in general,

$$
|w(x,\,x_0)|\leqq\left[mMN+\frac{PB}{M}\,|x-x_0|+Q|x-x_0|^2\right]^n\frac{W|x-x_0|^n}{n!}\,.
$$

But the expression on the right has the limit zero as n becomes indefinitely large. Therefore there is one and only one solution of (2) and hence of (1)[1] under the conditions of the problem.

5. The Integral Equation

$$
\begin{aligned}
u(x,\,x_0)&=g\left\{x,\,\int_{x_0}^{x}f_1\left[x,\,t,\,\int_{x_0}^{t}u(t,\,\xi)d\xi\right]u(t,\,x_0)dt,\right.\\
&\qquad\left.\cdots,\quad\int_{x_0}^{x}f_m\left[x,\,t,\,\int_{x_0}^{t}u(t,\,\xi)d\xi\right]u(t,\,x_0)dt\right\}.
\end{aligned}
\tag{1}
$$

THEOREM: *Let* $g\{x,\,v_1,\,v_2,\,\cdots,\,v_m\}$ *be continuous and satisfy, for its arguments* $v_1,\,v_2,\,\cdots,\,v_m$, *the Lipschitz condition*

$$
|g\{x,\,v_1+\mu_1,\,v_2+\mu_2,\,\cdots,\,v_m+\mu_m\}-g\{x,\,v_1,\,v_2,\,\cdots,\,v_m\}|\leqq A\sum_{k=1}^{m}|\mu_k|
$$

in the region $|x-x_0|\leqq\alpha$, $|v_1|\leqq\beta$, $|v_2|\leqq\beta$, $\cdots$, $|v_m|\leqq\beta$; *furthermore let*

$$
|g\{x,\,v_1,\,v_2,\,\cdots,\,v_m\}|\leqq N.
$$

Let $f_1[x,\,t,\,\theta]$, $f_2[x,\,t,\,\theta]$, $\cdots$, $f_m[x,\,t,\,\theta]$ *be continuous and satisfy, for the argument* θ, *the Lipschitz conditions*

$$
|f_k[x,\,t,\,\theta+\nu]-f_k[x,\,t,\,\theta]|\leqq B|\nu|\quad(k=1,\,2,\,\cdots,\,m),
$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - g\{x, 0, 0, \cdots, 0\}| \leqq \gamma$; *furthermore let*

$$|f_k[x, t, \theta]| \leqq M \qquad (k = 1, 2, \cdots, m).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{MN},^*$$

there exists, in $|x - x_0| \leqq \rho$, *one and only one continuous solution of the equation* (1).

Let us write (1) in the form

$$(1)^1 \qquad u(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D)u]].$$

Define

$$u_0(x, x_0) = [x, 0, 0, \cdots, 0],$$

$$u_1(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_0)u_0]],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u_n(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_{n-1})u_{n-1}]],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

where the conditions

$$|u_n(x, x_0) - [x, 0, 0, \cdots, 0]| \leqq \gamma$$

and

$$|\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_n)u_n]| \leqq \beta \qquad (n = 1, 2, \cdots)$$

are satisfied as may easily be seen by methods similar to those used in previous sections. Thus each successive approximation is justified.

To show the uniform convergence of the sequence of functions, defined above, to a definite limit which is, as seen by the forms of the approximations, a solution of the equation $(1)^1$ or (1), we write,

$$u_n(x, x_0) - u_{n-1}(x, x_0) = U_n(x, x_0) \quad (n = 1, 2, \cdots),$$

so that

$$u_n(x, x_0) = u_0(x, x_0) + U_1(x, x_0) + \cdots + U_n(x, x_0) \quad (n = 1, 2, \cdots).$$

We have

---

[*] Notice that the condition $\rho \leqq \gamma/mAMN$ is not introduced here, since it may be removed as in section 2.

$$|U_1| \leqq mANM|x - x_0|$$

$$|U_2| \leqq |[x, \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u_1]] - [x, \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_0)u_0]]|$$

$$\leqq A\{|\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u_1] - \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u_0]|$$

$$+ |\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u_0] - \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_0)D_0]|\}$$

$$\leqq A\left\{\left|\int_{x_0}^{x} |[u_1(t, x_0) - u_0(t, x_0)]\overset{*}{[}x, t; D_1]|dt\right|\right.$$

$$\left. + \left|\int_{x_0}^{x} |\overset{*}{[}x, t; D_1] - \overset{*}{[}x, t; D_0]u_0|dt\right|\right\}$$

$$\leqq m^2A^2MN\frac{|x - x_0|^2}{2!}[M + BN|x - x_0|],$$

$$|U_3| \leqq m^3A^3MN\frac{|x - x_0|^3}{3!}[M + BN|x - x_0|]^2,$$

. . . .

$$|U_n| \leqq m^nA^nMN\frac{|x - x_0|^n}{n!}[M + BN|x - x_0|]^{n-1},$$

. . . .

and the series

$$u_0 + mAMN\rho + m^2A^2MN\frac{\rho^2}{2!}[M + BN\rho] + \cdots$$

$$+ m^nA^nMN\frac{\rho^n}{n!}[M + BN\rho]^{n-1} + \cdots$$

being obviously convergent, we have

$$\operatorname*{Lim}_{n=\infty} u_n(x, x_0) = u(x, x_0)$$

uniformly for $|x - x_0| \leqq \rho$. This limit evidently satisfies $(1)^1$ and (1).

That there is only one such solution may be seen as follows. Suppose there were two, $u(x, x_0)$ and $v(x, x_0)$, and call the difference

$$w(x, x_0) = u(x, x_0) - z(x, x_0).$$

Then

$$|w(x, x_0)| \leqq W,$$

and we have

$$|w(x, x_0)| \leqq A\{|\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u] - \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_2)v]|\}$$

$$\leqq A\{|\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)u] - \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)v]|$$

$$+ |\overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_1)v] - \overset{*}{[}(x, t \overset{x}{\underset{x_0}{;}} D_2)v]|\},$$

where

$$D_1 = \int_{x_0}^{t} u(t,\ \xi)d\xi \qquad \text{and} \qquad D_2 = \int_{x_0}^{t} v(t,\ \xi)d\xi.$$

Finally we have

$$(a)\quad \begin{aligned} |w(x,\ x_0)| &\leqq A\left\{mM\left|\int_{x_0}^{x}|w(t,\ x_0)|dt\right| + mBN\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}|w(t,\ \xi)|d\xi\right|dt\right|\right\} \\ &\leqq A\left\{mMW|x - x_0| + mNBW\frac{|x - x_0|^2}{2!}\right\}; \end{aligned}$$

substituting this last result in (a) we have

$$|w(x,\ x_0)| \leqq m^2A^2M^2W\frac{|x - x_0|^2}{2!} + 2m^2A^2MBNW\frac{|x - x_0|^3}{3!}$$

$$+ m^2A^2MNBW\frac{|x - x_0|^4}{4!} \leqq m^2A^2W\frac{|x - x_0|^2}{2!}[M + NB|x - x_0|]^2.$$

Repeating this process, we get

$$|w(x,\ x_0)| \leqq m^nA^nW\frac{|x - x_0|^n}{n!}[M + NB|x - x_0|]^n,$$

which, as n becomes indefinitely large, approaches zero as a limit, therefore $w(x,\ x_0) = 0$. Consequently

$$u(x,\ x_0) = z(x,\ x_0).$$

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